Lowen, W. (2008) "Hochschild Cohomology of Presheaves as Map-Graded Categories,"
International Mathematics Research Notices, Vol. 2008, Article ID rnn118, 32 pages.
doi:10.1093/imrn/rnn118

Hochschild Cohomology of Presheaves as Map-Graded Categories

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Building on the work of Gerstenhaber and Schack, we define the Hochschild complex of a presheaf (or a pseudofunctor) $\mathcal{A}$ on a category $\mathcal{U}$ as the Hochschild complex of an associated "$\mathcal{U}$-graded" category, motivated by the classical correspondence between pseudofunctors and fibered categories. We show that this complex is a B_∞-algebra which controls the deformation theory of $\mathcal{A}$ *as a pseudofunctor*.

1 Introduction

In [2–4], Gerstenhaber and Schack study deformation theory and Hochschild cohomology of presheaves of algebras. For a presheaf $\mathcal{A}$, these authors define a Hochschild complex $\mathbf{C}_G(\mathcal{A})$ and they construct a single algebra A for which they prove the existence of isomorphisms of Hochschild cohomology $HH^n_G(\mathcal{A}) \cong HH^n(A)$ [3]. Unlike in the case of algebras, there is no 1-1 correspondence between the first-order deformations of a presheaf and its second Hochschild cohomology.

This paper is the first one in a broader project, where the intention is to put the results of Gerstenhaber and Schack in a different perspective, and extend them, by introducing map-graded categories and their Hochschild cohomology. Map-graded

Received May 23, 2008; Revised May 23, 2008; Accepted August 27, 2008

categories can be regarded as "group-graded algebras with several objects," where the word "map" refers to the morphisms in an underlying base category $\mathcal{U}$. Our original interest in Hochschild cohomology for map-graded categories stems from its role in the existence of a local to global spectral sequence for Hochschild cohomology of general ringed spaces, a result to be proved in a subsequent paper (the proof of a special case can be found in [9]).

In the first part of this paper, we work out the algebraic correspondence between pseudofunctors on $\mathcal{U}$ with values in k-linear categories, and fibered $\mathcal{U}$-graded categories. This is a linear version of the classical correspondence between pseudofunctors and fibered categories [1].

Map-graded categories are algebraically very close to ordinary algebras. For a $\mathcal{U}$-graded category $\mathfrak{a}$, we define a Hochschild complex $\mathbf{C}(\mathfrak{a})$ that shares all the "higher structure" with the Hochschild complex of an algebra: it is a B_∞-algebra [6]. As such, it controls a natural deformation theory of $\mathfrak{a}$.

The main point that we want to make is that for a presheaf, or more general pseudofunctor, $\mathcal{A}$, putting $\mathbf{C}(\mathcal{A}) = \mathbf{C}(\mathfrak{a})$ where $\mathfrak{a}$ is the $\mathcal{U}$-graded category associated to $\mathcal{A}$, is a sensible definition for the Hochschild complex of $\mathcal{A}$. The $\mathcal{U}$-graded category $\mathfrak{a}$ can be seen as a variant of Gerstenhaber and Schack's algebra A. When the category $\mathcal{U}$ is a poset, $\mathfrak{a}$ can equivalently be described as a linear category in which $\mathfrak{a}(A_V, A_U) = 0$ unless $V \subset U$ (this is the point of view taken in [8–10]). While these zeros do not affect the deformation theory, they precisely reduce the Hochschild complex to the "$\mathcal{U}$-graded" Hochschild complex. When $\mathcal{U}$ is moreover finite, then this linear category $\mathfrak{a}$ can equivalently be described as a matrix algebra, and then this is precisely the algebra A. However, for a general $\mathcal{U}$, Gerstenhaber and Schack first doubly subdivide $\mathcal{U}$ in order to obtain a poset, and then look at the algebra of the induced presheaf on this poset to arrive at the algebra A. They end up with

- the complex $\mathbf{C}_G(\mathcal{A})$, which lacks the beautiful structure of $\mathbf{C}(A)$, for example, it fails to be a dg Lie algebra;
- the quasi-isomorphic complex $\mathbf{C}(A)$, which is a B_∞-algebra, but which is associated to an object A which is already relatively "far" from the original presheaf.

For this reason, we propose (the quasi-isomorphic) $\mathbf{C}(\mathfrak{a})$ as a better candidate Hochschild complex of $\mathcal{A}$. Thanks to the correspondence between pseudofunctors and fibered map-graded categories, one could argue that $\mathfrak{a}$ really *is* $\mathcal{A}$ in a sense, and in the second part of the paper we prove that $\mathbf{C}(\mathfrak{a})$ explicitly controls the deformation theory of $\mathcal{A}$ *as a* *pseudofunctor*. This explains why the strict presheaf deformations considered by Gerstenhaber and Schack are not enough to understand the entire Hochschild cohomology.

2 Map-Graded Categories, Fibered Map-Graded Categories, and Pseudofunctors

Let $\mathcal{U}$ be a small category and k a commutative ring. In this section, we introduce $\mathcal{U}$-graded categories and we give a detailed account of a k-linear version of the classical correspondence between pseudofunctors and fibered categories [1] (see also [11]).

2.1 Map-graded categories

Definition 2.1. A $\mathcal{U}$-graded k-quiver $\mathfrak{a}$ consists of the following data: for every object $U \in \mathcal{U}$, we have a set of objects $\mathfrak{a}_U$, and for every morphism $u : V \longrightarrow U$ in $\mathcal{U}$ and objects $A \in \mathfrak{a}_V$, $B \in \mathfrak{a}_U$, we have a k-module $\mathfrak{a}_u(A, B)$. A precomposition μ on $\mathfrak{a}$ consists of operations

$$\mu^{u,v,A,B,C} : \mathfrak{a}_u(B, C) \otimes \mathfrak{a}_v(A, B) \longrightarrow \mathfrak{a}_{uv}(A, C).$$

A preidentity id on $\mathfrak{a}$ consists of elements

$$id^A \in \mathfrak{a}_1(A, A).$$

We say that

(1) $(\mathfrak{a}, \mu)$ is a $\mathcal{U}$-graded associative k-quiver (and μ is a composition on $\mathfrak{a}$) if the relations

$$\mu^{w,uv,A,C,D}\big(\mu^{u,v,A,B,C} \otimes 1_{\mathfrak{a}_w(C,D)}\big) = \mu^{wu,v,A,B,D}\big(1_{\mathfrak{a}_v(A,B)} \otimes \mu^{w,u,B,C,D}\big)$$

are satisfied;

(2) $(\mathfrak{a}, \mu, id)$ is a $\mathcal{U}$-graded k-quiver with identity (and id is an identity on $(\mathfrak{a}, \mu)$) if the relations

$$\mu^{u,1,A,A,B}\big(1_{\mathfrak{a}_u(A,B)} \otimes id^A\big) = 1_{\mathfrak{a}_u(A,B)} = \mu^{1,u,A,B,B}\big(id^B \otimes 1_{\mathfrak{a}_u(A,B)}\big)$$

are satisfied;

(3) $(\mathfrak{a}, \mu, id)$ is a $\mathcal{U}$-graded k-category if $\mathfrak{a}$ is a $\mathcal{U}$-graded associative k-quiver with identity. □

Throughout, we will freely suppress sub- and superscripts whenever they are clear from the context. For example, we will formulate condition (1) simply as $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$ and condition (2) simply as $\mu(1 \otimes id) = 1 = \mu(id \otimes 1)$.

Remark 2.2. If $\mathcal{U}$ has a single object $*$ and $\mathfrak{a}_*$ has a single object $*'$, then $G = \mathcal{U}(*, *)$ is a monoid and a $\mathcal{U}$-graded k-category$(\mathfrak{a}, \mu, id)$ corresponds to a G-graded k-algebra A with $A_g = \mathfrak{a}_g(*', *')$. Hence, map-graded categories are multiobject versions of monoid-graded rings. □

Definition 2.3. A prefibered structure δ on a $\mathcal{U}$-graded quiver $\mathfrak{a}$ consists of elements

$$\delta^{u,A} \in \mathfrak{a}_u(\delta^* A, A)$$

for $u : V \longrightarrow U$ and $A \in \mathfrak{a}_U$, where the objects $\delta^* A \in \mathfrak{a}_V$ are part of the structure. Let μ be a precomposition on a $\mathcal{U}$-graded quiver $\mathfrak{a}$. A morphism $\delta \in \mathfrak{a}_u(B, C)$ for $u : V \longrightarrow U$ is called cartesian if for every $v : W \longrightarrow V$ and $A \in \mathfrak{a}_W$, the map

$$\mu(\delta, -) : \mathfrak{a}_v(A, B) \longrightarrow \mathfrak{a}_{uv}(A, C)$$

is an isomorphism of k-modules. We say that

(1) $(\mathfrak{a}, \mu, \delta)$ is a fibered $\mathcal{U}$-graded k-quiver if the morphisms $\delta^{u,A}$ are cartesian.
(2) $(\mathfrak{a}, \mu, id, \delta)$ is a fibered $\mathcal{U}$-graded k-category if it is both fibered, and a $\mathcal{U}$-graded k-category. □

If no confusion arises, we will drop the mention of k from our terminology, for example, we will speak about $\mathcal{U}$-graded categories rather than $\mathcal{U}$-graded k-categories.

Definition 2.4. Let $(\mathfrak{a}, \mu, id)$ be a $\mathcal{U}$-graded category. A morphism $\rho \in \mathfrak{a}_1(A, B)$ is called an isomorphism if there exists a $\rho' \in \mathfrak{a}_1(B, A)$ with $\mu(\rho', \rho) = id^A$ and $\mu(\rho, \rho') = id^B$. □

If ρ is an isomorphism, then a ρ' like in the definition is necessarily unique and will be called the inverse ρ^{-1} of ρ.

The following lemma shows that on a map-graded category, a fibered structure is, if it exists, unique up to canonical isomorphism.

Lemma 2.5. Let $(\mathfrak{a}, \mu, id)$ be a map-graded category. Suppose $\delta_1 \in \mathfrak{a}_u(B, C)$ and $\delta_2 \in \mathfrak{a}_u(A, C)$ are cartesian morphisms. There is a unique morphism $\rho \in \mathfrak{a}_1(A, B)$ with $\delta_1\rho = \delta_2$ and ρ is an isomorphism. □

Proof. Easy. ■

Definition 2.6. Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathcal{U}$-graded quivers. A quiver morphism $\varphi : \mathfrak{a} \longrightarrow \mathfrak{b}$ consists of maps $\varphi^U : \mathfrak{a}_U \longrightarrow \mathfrak{b}_U$ and morphisms

$$\varphi^{u,A,B} : \mathfrak{a}_u(A, B) \longrightarrow \mathfrak{b}_u(\varphi A, \varphi B)$$

for every $u : V \longrightarrow U$, $A \in \mathfrak{a}_V$, and $B \in \mathfrak{a}_U$. If $(\mathfrak{a}, \mu, id)$ and $(\mathfrak{b}, \mu', id')$ are $\mathcal{U}$-graded categories, then we call φ a morphism of $\mathcal{U}$-graded categories if the following conditions hold:

(1) $\varphi\mu = \mu'(\varphi \otimes \varphi)$;
(2) $\varphi(id) = id'$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are moreover fibered, then a fibered quiver morphism contains additional isomorphisms

$$\gamma^{u,A} \in \mathfrak{b}_1(\delta^{'*}\varphi(A), \varphi(\delta^* A)),$$

and (φ, γ) is a fibered morphism of $\mathcal{U}$-graded categories if, in addition to (1) and (2),

(3) $\mu'(\varphi(\delta^{u,A}), \gamma^{u,A}) = \delta'^{u,\varphi A}$

holds. □

These notions give rise to categories Mapgr of $\mathcal{U}$-graded categories and morphisms, and Fibgr of fibered $\mathcal{U}$-graded categories and fibered morphisms. Let $\mathsf{Mapgr}_{\mathrm{iso}} \subseteq \mathsf{Mapgr}$ and $\mathsf{Fibgr}_{\mathrm{iso}} \subseteq \mathsf{Fibgr}$ be the groupoids containing only the isomorphism of the ambient categories.

Proposition 2.7. The natural functor $\mathsf{Fibgr}_{\mathrm{iso}} \longrightarrow \mathsf{Mapgr}_{\mathrm{iso}}$ is fully faithful. □

Proof. Consider fibered $\mathcal{U}$-graded categories $(\mathfrak{a}, \mu, id, \delta)$ and $(\mathfrak{b}, \mu', id', \delta')$ and an isomorphism $\varphi : \mathfrak{a} \longrightarrow \mathfrak{b}$ of $\mathcal{U}$-graded categories. This implies that $\delta^{u,A} \in \mathfrak{a}_u(\delta^* A, A)$ gets mapped to a cartesian morphism $\varphi(\delta^{u,A}) \in \mathfrak{b}_u(\varphi(\delta^* A), \varphi(A))$. On the other hand, $\delta'^{u,\varphi(A)} \in \mathfrak{b}_u(\delta'^* \varphi(A), \varphi(A))$ is cartesian too, whence, by Lemma 2.5, there is a unique isomorphism $\gamma^{u,A}$ satisfying condition (3) of Definition 2.6. ■

It will be convenient to consider fibered $\mathcal{U}$-graded categories in which for every $A \in \mathfrak{a}_U$, we have $\delta^{1_U,*} A = A$ and $\delta^{1_U,A} = id^A \in \mathfrak{a}_1(A, A)$. We denote the full subcategory of these fibered-graded categories by $\mathsf{Fibgr}_0 \subseteq \mathsf{Fibgr}$. The following proposition shows that they form a skeletal subcategory.

Proposition 2.8. Let $(\mathfrak{a}, \mu, id)$ be a map-graded category. If there exists a fibered structure on $\mathfrak{a}$, then any two fibered structures δ_1 and δ_2 yield canonically isomorphic fibered-graded categories $(\mathfrak{a}, \mu, id, \delta_1)$ and $(\mathfrak{a}, \mu, id, \delta_2)$. In particular, $(\mathfrak{a}, \mu, id, \delta)$ is canonically isomorphic to $(\mathfrak{a}, \mu, id, \tilde{\delta}) \in \mathsf{Fibgr}_0$ where we have only changed the $\delta^{1,A}$'s into 1_A's. □

Proof. This immediately follows from Proposition 2.7, since the identity is an isomorphism of $\mathcal{U}$-graded categories. ■

2.2 Bimodules over map-graded categories

Map-graded categories are in many respects similar to ordinary linear categories. However, to obtain a natural notion of modules, we need two of them.

Definition 2.9. Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathcal{U}$-graded categories. An $\mathfrak{a}$-$\mathfrak{b}$-bimodule M consists of k-modules

$$M_u(A, B)$$

for $u : V \longrightarrow U$, $A \in \mathfrak{a}_V$, $B \in \mathfrak{b}_U$ and morphisms

$$\rho : \mathfrak{b}_u(C, D) \otimes M_v(B, C) \otimes \mathfrak{a}_w(A, B) \longrightarrow M_{uvw}(A, D),$$

satisfying the following associativity and identity conditions:

(1) $\rho(\mu \otimes 1 \otimes \mu) = \rho(1 \otimes \rho \otimes 1)$;
(2) $\rho(id \otimes 1 \otimes id) = 1$.

A morphism $f : M \longrightarrow N$ of bimodules consists of morphisms

$$M_u(A, B) \longrightarrow N_u(A, B)$$

of k-modules, satisfying $f\rho_M = \rho_N(1 \otimes f \otimes 1)$. $\square$

We obtain a category $\mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{a}$-$\mathfrak{b}$ bimodules. As usual, we denote $\mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a}) = \mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a}, \mathfrak{a})$ and call the objects $\mathfrak{a}$-*bimodules*. For ordinary linear categories $\mathfrak{a}$ and $\mathfrak{b}$, $\mathfrak{a}$-$\mathfrak{b}$-bimodules can equivalently be described as modules over the tensor product linear category $\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}$. Remarkably, two $\mathcal{U}$-*graded* categories $\mathfrak{a}$ and $\mathfrak{b}$ have an associated "tensor product" $\mathfrak{a}^{\mathrm{op}} \otimes_{\mathcal{U}} \mathfrak{b}$ which is a *linear* category.

Definition 2.10. Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathcal{U}$-graded categories. The tensor product $\mathfrak{a}^{\mathrm{op}} \otimes_{\mathcal{U}} \mathfrak{b}$ is the k-linear category with

$$\mathrm{Ob}\big(\mathfrak{a}^{\mathrm{op}} \otimes_{\mathcal{U}} \mathfrak{b}\big) = \coprod_{u:V\longrightarrow U} (\mathfrak{a}_V \times \mathfrak{b}_U)$$

and

$$\mathrm{Hom}((u, A, B), (u', A', B')) = \oplus_{u'=zuv}\mathfrak{a}_v(A', A) \otimes \mathfrak{b}_z(B, B'). \qquad \square$$

Proposition 2.11. There is an isomorphism of linear categories

$$\mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a}, \mathfrak{b}) \cong \mathsf{Mod}(\mathfrak{a} \otimes_{\mathcal{U}} \mathfrak{b}). \qquad \square$$

Proof. This is almost a tautology. $\blacksquare$

2.3 Pseudofunctors

Let $\mathcal{U}$ be a small category and k a commutative ring as before.

Definition 2.12. A lax k-quiver $\mathcal{A}$ over a category $\mathcal{U}$ consists of the following data. For every object $U \in \mathcal{U}$, we have a k-quiver $\mathcal{A}(U)$ (i.e. a set of objects $\mathrm{Ob}(\mathcal{A}(U))$, and for any two objects A, B, a k-module $\mathcal{A}(U)(A, B)$). Furthermore, for every map $u : V \longrightarrow U$ in $\mathcal{U}$, we have a map $u^* : \mathrm{Ob}(\mathcal{A}(U)) \longrightarrow \mathrm{Ob}(\mathcal{A}(V))$.

A prelax structure $l = (m, f, c)$ on $\mathcal{A}$ consists of the following operations.

- For $A = (A_0, A_1, A_2) \in \mathrm{Ob}(\mathcal{A}(U))^3$, there is a 2-ary operation

$$m^{U,A} \in \mathrm{Hom}_k(\mathcal{A}(U)(A_1, A_2) \otimes \mathcal{A}(U)(A_0, A_1), \mathcal{A}(U)(A_0, A_2)).$$

- For $A = (A_0, A_1) \in \mathrm{Ob}(\mathcal{A}(U))^2$ and $u : V \longrightarrow U$, there is a 1-ary operation

$$f^{u,A} \in \mathrm{Hom}_k(\mathcal{A}(U)(A_0, A_1), \mathcal{A}(V)(u^* A_0, u^* A_1)).$$

- For $A \in \mathcal{A}(U)$ and $\sigma = (u : V \longrightarrow U, v : W \longrightarrow V)$, there is a 0-ary operation

$$c^{\sigma,A} \in \mathcal{A}(W)(v^* u^* A, (uv)^* A)).$$

A global preidentity z on $\mathcal{A}$ consists of a 0-ary operation

$$z^A \in \mathcal{A}(U)(A, 1^* A)$$

for every $A \in \mathcal{A}(U)$. A local preidentity 1 on $\mathcal{A}$ consists of a 0-ary operation

$$1^A \in \mathcal{A}(U)(A, A)$$

for every $A \in \mathcal{A}(U)$. We say that

(1) $(\mathcal{A}, l)$ is an associative lax quiver (and l is a lax structure on $\mathcal{A}$) if the following relations hold:
  (a) for $A_0, A_1, A_2, A_3 \in \mathrm{Ob}(\mathcal{A}(U))$, there is a relation

$$m^{A_{0,2,3}}(1 \otimes m^{A_{0,1,2}}) = m^{A_{0,1,3}}(m^{A_{1,2,3}} \otimes 1);$$

  (b) for $A_0, A_1, A_2 \in \mathcal{A}(U)$ and $u : V \longrightarrow U$, there is a relation

$$m^{u^* A_0, u^* A_1, u^* A_2}(f^{u,A_{1,2}} \otimes f^{u,A_{0,1}}) = f^{u,A_{0,2}} m^{A_{0,1,2}};$$

  (c) for $A_0, A_1 \in \mathcal{A}(U)$ and $\sigma = (u : V \longrightarrow U, v : W \longrightarrow V)$, there is a relation

$$m^{v^* u^* A_0, v^* u^* A_1, (uv)^* A_1}(c^{\sigma,A_1} \otimes (f^{v,u^* A_{0,1}} f^{u,A_{0,1}})$$
$$= m^{v^* u^* A_0, (uv)^* A_0, (uv)^* A_1}(f^{uv,A_{0,1}} \otimes c^{\sigma,A_0});$$

(d) for $A \in \mathcal{A}(U)$ and $\sigma = (u : V \longrightarrow U, v : W \longrightarrow V, w : Z \longrightarrow W)$, there is a relation

$$m^{w^*v^*u^*A, w^*(uv)^*, (uvw)^*}(c^{uv,w,A_0} \otimes f^{w,v^*u^*A,(uv)^*A}(c^{u,v,A_0}))$$
$$= m^{w^*v^*u^*A,(vw)^*u^*A,(uvw)^*A}(c^{u,vw} \otimes c^{v,u,u^*A_0}).$$

(2) $(\mathcal{A}, l, z)$ is a lax quiver with global identity (and z is a global identity on $(\mathcal{A}, l)$) if the following relations hold:

(a) for $A, B \in \mathcal{A}(U)$,

$$m^{A,1^*A,1^*B}(f^{1,A,B} \otimes z^A) = m^{A,B,1^*B}(z^B \otimes 1);$$

(b) for $A \in \mathcal{A}(U)$,

$$m^{u^*A,1^*u^*A,u^*A}(c^{u,1,A}, z^{u^*A}) = 1^{u^*A} = m^{u^*A,u^*1^*A,u^*A}(c^{1,u,A}, f^{u,A,1^*A}(z^A)).$$

(3) $(\mathcal{A}, l, 1)$ is a lax quiver with local identity (and 1 is a local identity on $(\mathfrak{a}, l)$) if the following relations hold:

(a) for $A, B \in \mathcal{A}(U)$,

$$m^{A,A,B}(1 \otimes 1^A) = 1 = m^{A,B,B}(1^B \otimes 1);$$

(b) for $A \in \mathcal{A}(U)$ and $u : V \longrightarrow U$,

$$f^{u,A,A}(1^A) = 1^{u^*A}.$$

(4) $(\mathcal{A}, l, z, 1)$ is a lax functor if it is an associative lax quiver with local and global identities.

If a lax quiver $\mathcal{A}$ is endowed with a local preidentity 1, a map $a : A \longrightarrow A'$ in $\mathcal{A}(U)$ is called an isomorphism if there is a map $a' : A' \longrightarrow A$ with $m(a', a) = 1^A$, $m(a, a') = 1^{A'}$. In all the above terminology, we replace the word lax by pseudo if the morphisms $c^{\sigma,A}$ and z^A that occur, are isomorphisms. □

Remark 2.13. In the above definition, 1(a) combined with (3) expresses that the operations m and 1 make the quivers $\mathcal{A}(U)$ into categories. If 1(a) and (3) hold, then 1(b) combined with 3(b) expresses that the operations f define functors $u^* = f^u : \mathcal{A}(U) \longrightarrow \mathcal{A}(V)$.

If 1(a), 1(b), and (3) hold, then 1(c) expresses that the operations c define natural transformations $v^*u^* \longrightarrow (uv)^*$. If 1(a), 1(b), and (3) hold, then 2(a) expresses that z defines natural transformations $z^U : 1_{\mathcal{A}(U)} \longrightarrow (1_U)^*$. □

Example 2.14. A *presheaf of categories* on $\mathcal{U}$ is a pseudofunctor $\mathcal{A}$ on $\mathcal{U}$ with $(uv)^* = v^*u^*$ and $c^{u,v,A} = 1_{v^*u^*A}$. □

Definition 2.15. Let $\mathcal{A}$ and $\mathcal{B}$ be lax quivers over $\mathcal{U}$. A morphism of lax quivers $(g,\alpha) : \mathcal{A} \longrightarrow \mathcal{B}$ consists of the following data:

- maps $g^U : \mathrm{Ob}(\mathcal{A}(U)) \longrightarrow \mathrm{Ob}(\mathcal{B}(U))$;
- for $A, B \in \mathcal{A}(U)$, maps

$$g^{A,B} : \mathcal{A}(U)(A,B) \longrightarrow \mathcal{B}(U)(gA, gB);$$

- for $A \in \mathcal{A}(U)$, $u : V \longrightarrow U$, elements

$$\alpha^{u,A} \in \mathcal{B}(V)\big(u^*_{\mathcal{B}} g^U(A), g^V u^*_{\mathcal{A}}(A)\big).$$

If $(\mathcal{A}, l, z, 1)$ and $(\mathcal{B}, l', z', 1')$ are pseudofunctors, then we call (g,α) a morphism of pseudofunctors if the $\alpha^{u,A}$ are isomorphisms and the following conditions hold:

(1) $gm = m'(g \otimes g)$;
(2) $m'(gf, \alpha) = m'(\alpha, f'g)$;
(3) $m'(\alpha, c') = m'(g(c), \alpha, \alpha)$;
(4) $g(1) = 1'$;
(5) $m'(\alpha^1, z') = g(z)$. □

These notions give rise to a category Psfun of pseudofunctors and their morphisms.

Remark 2.16.

(1) Conditions (1) and (2) express functoriality of $g^U : \mathcal{A}(U) \longrightarrow \mathcal{B}(U)$. Condition (3) expresses naturality of $\alpha : f'g \longrightarrow gf$.
(2) Morphisms of pseudofunctors are usually called pseudonatural transformations in the literature. □

Lemma 2.17. A morphism $(g,\alpha) : \mathcal{A} \longrightarrow \mathcal{B}$ of pseudofunctors is an isomorphism in Psfun if and only if every component $g^U : \mathcal{A}(U) \longrightarrow \mathcal{B}(U)$ is an isomorphism of quivers, i.e. if it is an isomorphism on objects, and if every $g^{A,B} : \mathcal{A}(U)(A,B) \longrightarrow \mathcal{B}(U)(gA,gB)$ is an isomorphism of k-modules. $\square$

Proof. If the $g^U : \mathcal{A}(U) \longrightarrow \mathcal{B}(U)$ are functors and isomorphisms of quivers, we have inverse functors $(g^U)^{-1} : \mathcal{B}(U) \longrightarrow \mathcal{A}(U)$. To complete these into an inverse morphism (g^{-1},β), it suffices to transform $\alpha : f_{\mathcal{B}} g^U \longrightarrow g^V f_{\mathcal{A}}$ into $\beta^{-1} = g^{V^{-1}} \alpha g^{U^{-1}}$. $\blacksquare$

Lemma 2.18. Consider a pseudofunctor $(\mathcal{B}, l', z', 1')$ and a lax quiver $\mathcal{A}$. Suppose we are given a morphism $(g,\alpha) : \mathcal{A} \longrightarrow \mathcal{B}$ of quivers such that

- the $g^U : \mathcal{A}(U) \longrightarrow \mathcal{B}(U)$ are isomorphisms of quivers;
- the $\alpha^{u,A}$ are isomorphisms in $\mathcal{B}(V)$.

There is a unique structure $(l, z, 1)$ on $\mathcal{A}$ such that

- $(\mathcal{A}, l, z, 1)$ is a pseudofunctor;
- $(g,\alpha) : \mathcal{A} \longrightarrow \mathcal{B}$ is an isomorphism of pseudofunctors. $\square$

Proof. There is an obvious way to transport the structure from $\mathcal{B}$ to $\mathcal{A}$, namely $m = g^{-1} m'(g \otimes g)$, $f = g^{-1} m'(\alpha, f' g, \alpha^{-1})$, $c = g^{-1} m'(\alpha, c'^{gA}, \alpha^{-1}, \alpha^{-1})$, $1 = g^{-1}(1'^{gA})$, and $z = g^{-1} m'(\alpha, z'^{gA})$. It is immediate to check that the conditions of Definitions 2.12 and 2.15 are fulfilled, and by Lemma 2.17, (g,α) becomes an isomorphism of pseudofunctors. $\blacksquare$

It will be convenient to consider lax quivers $(\mathcal{A}, 1, 1)$ with local preidentity 1 and with *trivial global preidentity* $z = 1$, i.e. such that

(1) for $1^U \in \mathcal{U}(U,U)$, we have $1^* = 1^{\mathrm{Ob}(\mathcal{A}(U))}$;
(2) $z^A = 1^A \in \mathcal{A}(U)(A,A)$.

We will denote the full subcategory of pseudofunctors with trivial global identity $z = 1$ by $\mathsf{Psfun}_0 \subseteq \mathsf{Psfun}$. The following proposition shows that they form a skeletal subcategory.

Proposition 2.19. Every pseudofunctor $(\mathcal{A}, l, z, 1)$ is canonically isomorphic to a pseudofunctor $(\tilde{\mathcal{A}}, \tilde{l}, \tilde{z}, \tilde{1}) \in \mathsf{Psfun}_0$ with $(\tilde{\mathcal{A}}(U), \tilde{m}, \tilde{1}) = (\mathcal{A}(U), m, 1)$. $\square$

Proof. We define the lax quiver $\tilde{\mathcal{A}}$ by $\tilde{\mathcal{A}}(U) = \mathcal{A}(U)$, $\tilde{u}^* = u^*$ for $1 \neq u$ and $\tilde{1}^{U,*} = 1^{\mathrm{Ob}(\mathcal{A}(U))}$. Next we define isomorphisms of quivers $g^U = 1^{\mathcal{A}(U)} : \tilde{\mathcal{A}}(U) \longrightarrow \mathcal{A}(U)$ and $\mathcal{A}(U)$-isomorphisms $\alpha^{u,A} = 1^{u^*A}$ for $1 \neq u$ and $\alpha^{1,A} = (z^A)^{-1} \in \mathcal{A}(U)(1^*A, A)$. According to Lemma 2.18, there is a unique structure $(\tilde{l}, \tilde{z}, \tilde{1})$ on $\tilde{\mathcal{A}}$ such that (g,α) becomes an isomorphism of pseudofunctors. Moreover, this structure has $\tilde{m} = m$, $\tilde{1} = 1$, $\tilde{f}^u = f^u$ for $1 \neq u$ and $\tilde{f}^1 = m((z^B)^{-1}, f, z^A) = 1$ and $\tilde{z} = m((z^A)^{-1}, z^A) = 1^A$. ■

Obviously, presheaves on $\mathcal{U}$ belong to Psfun_0.

2.4 Pseudofunctors versus fibered map-graded categories

There is a well-known correspondence between pseudofunctors and fibered categories going back to [1] (see also [11]). In this section, we investigate this more closely in the k-enriched setting.

Let $\mathcal{A}$ be a lax quiver on $\mathcal{U}$. We define an associated $\mathcal{U}$-graded quiver $\mathfrak{a} = \mathfrak{a}(\mathcal{A})$ by

$$\mathfrak{a}_u(A, B) = \mathcal{A}(V)(A, u^*B)$$

for $u : V \longrightarrow U$ in $\mathcal{U}$, $A \in \mathcal{A}(V)$ and $B \in \mathcal{A}(U)$.

First we will look at preidentities and precartesian structures. Suppose z is a global preidentity on $\mathcal{A}$. Then we define a preidentity id on $\mathfrak{a}$ by putting

$$id^A = z^A \in \mathcal{A}(U)(A, 1^*A) = \mathfrak{a}_1(A, A). \tag{1}$$

Clearly, this defines a 1-1 correspondence between global preidentities on $\mathcal{A}$ and preidentities on $\mathfrak{a}$.

Suppose 1 is a local preidentity on $\mathcal{A}$. Then we define a prefibered structure δ on $\mathfrak{a}$ by putting $\delta^{u,A,*}A = u^*A$ and

$$\delta^{u,A} = 1^{u^*A} \in \mathcal{A}(U)(u^*A, u^*A) = \mathfrak{a}_u(u^*A, A). \tag{2}$$

Conversely, to give a local preidentity on $\mathcal{A}$ we have to specify elements $1^A \in \mathcal{A}(U)(A, A)$ for every A, and this cannot be done starting from a prefibered structure alone. In fact, in order to define such a local preidentity on $\mathcal{A}$ starting from the prefibered structure on $\mathfrak{a}$, we need a global preidentity z in which the $z^A \in \mathcal{A}(U)(A, 1^*A)$ are isomorphisms with respect to a composition on $\mathcal{A}$.

Now consider a prelax structure $l = (m, f, c)$ on $\mathcal{A}$. We define a precomposition $\mu = \mu(l)$ on $\mathfrak{a}$ in the following way. We have to give maps

$$\mu : \mathfrak{a}_u(B, C) \otimes \mathfrak{a}_v(A, B) \longrightarrow \mathfrak{a}_{uv}(A, C),$$

i.e. maps

$$\mu : \mathcal{A}(V)(B, u^*C) \otimes \mathcal{A}(W)(A, v^*B) \longrightarrow \mathcal{A}(W)(A, (uv)^*C).$$

Consider

$$f : \mathcal{A}(V)(B, u^*C) \longrightarrow \mathcal{A}(W)(v^*B, v^*u^*C)$$

and

$$c \in \mathcal{A}(W)(v^*u^*C, (uv)^*C).$$

We put

$$\mu = m(c \otimes m(f \otimes 1)).$$

Proposition 2.20. If $(\mathcal{A}, l, z, 1)$ is a pseudofunctor, then $(\mathfrak{a}, \mu, id, \delta)$ is a fibered-graded category. □

Proof. To prove this, let us write down all the conditions on $\mathfrak{a}$ in terms of $\mathcal{A}$. For δ to be cartesian, the relevant map $\mu(\delta^{u,A}, -) : \mathcal{A}(W)(A', v^*u^*A) \longrightarrow \mathcal{A}(W)(A', (uv)^*A)$ is given by

$$\mu(\delta^{u,A}, a) = m(c, m(f^v(1_{u^*A}), a)) = m(c, a), \tag{3}$$

and since c is an isomorphism, so is the composition with c. The morphisms id are identities for μ if and only if the following identities hold true for every $u : V \longrightarrow U$, $A \in \mathcal{A}(V)$, $B \in \mathcal{B}(U)$, $a \in \mathfrak{a}_u(A, B) = \mathcal{A}(V)(A, u^*B)$:

$$c^{B,1,u} 1^*(a) z^A = a = c^{B,u,1} u^*(z^B) a. \tag{4}$$

In (2), we may equivalently replace the right-hand side equality by

$$c^{B,u,1} u^*(z^B) = 1_{u^*B}, \tag{5}$$

which is precisely the right-hand side of condition 2(b) in the global identity condition. Using the local identity condition, we can rewrite the left-hand side of (2) as

$$c^{B,1,u} z^{u^*B} = 1_{u^*B}, \tag{6}$$

which is precisely the left-hand side of condition 2(b). It remains to prove associativity of μ, which follows from an explicit straightforward computation with the definition of μ. ■

Let $\mathfrak{a}$ be a $\mathcal{U}$-graded quiver with a prefibered structure δ. We define a lax quiver $\mathcal{A} = \mathcal{A}(\mathfrak{a})$ by $\mathrm{Ob}(\mathcal{A}(U)) = \mathfrak{a}_U$,

$$\mathcal{A}(U)(A, A') = \mathfrak{a}_1(A, A')$$

and for $u : V \longrightarrow U$, $u^*A = \delta^*A$. Suppose id is a preidentity on $\mathfrak{a}$. Then

$$1^A = id^A \in \mathfrak{a}_1(A, A) = \mathcal{A}(U)(A, A) \tag{7}$$

defines a local preidentity on $\mathcal{A}$. Furthermore, the prefibered structure yields elements

$$\delta^{1,A} \in \mathfrak{a}_1(u^*A, A) = \mathcal{A}(U)(u^*A, A).$$

Now suppose $(\mathfrak{a}, \mu, id, \delta)$ is a *cartesian*-graded quiver with a precomposition and a preidentity. Then we have an isomorphism

$$\mu(\delta^{1,A}, -) : \mathfrak{a}_1(A, u^*A) \longrightarrow \mathfrak{a}_1(A, A) = \mathcal{A}(U)(A, A),$$

and we define a global preidentity on $\mathcal{A}$ by letting

$$z^A \in \mathfrak{a}_1(A, u^*A) = \mathcal{A}(U)(A, u^*A)$$

equal the inverse image of 1^A under this isomorphism. In other words, z^A is the unique morphism with

$$\mu(\delta^{1,A}, z^A) = id^A. \tag{8}$$

Furthermore, we can now define a prelax structure l on $\mathcal{A}$. We put

$$m^{A,B,C} = \mu^{1,1,A,B,C} : \mathfrak{a}_1(B,C) \otimes \mathfrak{a}_1(A,B) \longrightarrow \mathfrak{a}_1(A,C).$$

For $u : U \longrightarrow V$ and $A, B \in \mathcal{A}(U)$, we define $f^{u,A,B}$ to be the composition

$$\mathfrak{a}_1(A,B) \xrightarrow[\mu(-,\delta^{u,A})]{} \mathfrak{a}_u(u^*A, B) \xrightarrow[\mu(\delta^{u,B},-)^{-1}]{} \mathfrak{a}_1(u^*A, u^*B).$$

In other words, for $a \in \mathfrak{a}_1(A,B)$, $u^*(a) \in \mathfrak{a}_1(u^*A, u^*B)$ is the unique morphism for which

$$\mu(a, \delta^{u,A}) = \mu(\delta^{u,B}, u^*(a)). \tag{9}$$

Finally, for $u : V \longrightarrow U$, $v : W \longrightarrow V$, $A \in \mathcal{A}(U)$, we consider

$$\mathfrak{a}_1(v^*u^*A, v^*u^*A) \xrightarrow[\mu(\delta^{v,u^*A},-)]{} \mathfrak{a}_v(v^*u^*A, u^*A) \xrightarrow[\mu(\delta^{u,A},-)]{} \mathfrak{a}_{uv}(v^*u^*A, A)$$

and

$$\mu(\delta^{uv,A}, -) : \mathfrak{a}_1(v^*u^*A, (uv)^*) \longrightarrow \mathfrak{a}_{uv}(v^*u^*A, A),$$

and put

$$c^{u,v,A} = \mu(\delta^{uv,A}, -)^{-1}(\mu(\delta^{u,A}, \mu(\delta^{v,u^*A}, id^{v^*u^*A}))).$$

In other words, $c^{u,v,A} \in \mathfrak{a}_1(v^*u^*A, (uv)^*A)$ is the unique morphism for which

$$\mu(\delta^{uv,A}, c^{u,v,A}) = \mu(\delta^{u,A}, \mu(\delta^{v,u^*A}, id^{v^*u^*A})). \tag{10}$$

Proposition 2.21. If $(\mathfrak{a}, \mu, id, \delta)$ is a fibered-graded category, then $(\mathcal{A}, l, z, 1)$ is a pseudo-functor. □

Proof. Let us prove first that z is a global identity for $\mathcal{A}$. Consider $b : C \longrightarrow D$ in $\mathcal{A}(U)$. We have to show that

$$m(1^*(b), z) = m(z, b).$$

We compute $\mu(\delta, \mu(1^*(b), z)) = \mu(\mu(\delta, 1^*(b)), z) = \mu(\mu(b, \delta), z) = \mu(b, \mu(\delta, z)) = \mu(b, id) = b = \mu(\delta, \mu(z, b))$, whence the desired equality follows, since δ is cartesian.

Next we have to show that

$$m(c, z) = 1 = m(c, u^* z).$$

We compute that $\mu(\delta, \mu(c, z)) = \mu(\delta, \mu(\delta, z)) = \mu(\delta, 1) = \delta$ and that $\mu(\delta, \mu(c, u^* z)) = \mu(\delta, \mu(\delta, u^* z)) = \mu(\mu(\delta, z), \delta) = \delta$, whence the result follows.

Let us prove that 1 is a local identity for $\mathcal{A}$. Condition 3(a) is immediate and 3(b), follows since $\mu(\delta, u^*(1)) = \mu(1, \delta) = \mu(\delta, 1)$.

Finally, we have to prove the list of associativity conditions. Condition 1(a) is immediate. For 1(b), we compute $\mu(\delta, u^*(a), u^*(b)) = \mu(a, b, \delta) = \mu(\delta, u^*(\mu(a, b)))$. For 1(c), we compute $\mu(\delta, c, v^* u^*(a)) = \mu(\delta, \delta, v^* u^*(a)) = \mu(\delta, u^*(a), \delta) = \mu(a, \delta, \delta)$ and $\mu(\delta, (uv)^*(a), c) = \mu(a, \delta, c) = \mu(a, \delta, \delta)$. For 1(d), we compute $\mu(\delta, c, w^*(c)) = \mu(\delta, \delta, w^*(c)) = \mu(\delta, c, \delta) = \mu(\delta, \delta, \delta)$ and $\mu(\delta, c, c) = \mu(\delta, \delta, \delta)$. ■

Theorem 2.22. The above correspondence on objects extends to an *equivalence* of categories

$$\mathsf{Psfun} \cong \mathsf{Fibgr}$$

that restricts to an *isomorphism* of categories

$$\mathsf{Psfun}_0 \cong \mathsf{Fibgr}_0$$

between the skeletal subcategories. □

Proof. Let us first explicitate the functor $\mathsf{Psfun} \longrightarrow \mathsf{Fibgr}$ on morphisms. Let $(g, \alpha) : \mathcal{A} \longrightarrow \mathcal{B}$ be a pseudonatural transformation. Put $\mathfrak{a} = \mathfrak{a}(\mathcal{A})$, $\mathfrak{b} = \mathfrak{b}(\mathcal{B})$. We are to define $(\varphi, \gamma) : \mathfrak{a} \longrightarrow \mathfrak{b}$. On objects, φ coincides with g. For $u : V \longrightarrow U$, $A \in \mathfrak{a}_V$, $B \in \mathfrak{a}_U$, the map

$$\varphi^{u,A,B} : \mathcal{A}(V)(A, u^* B) \longrightarrow \mathcal{B}(V)(gA, u^*(gB))$$

is given by

$$\varphi^{u,A,B} = m'(\alpha^{-1}, g).$$

Furthermore, we put

$$\gamma^{u,A} = m'(z', \alpha^{u,A}) \in \mathcal{B}(V)(u^*(gA), 1^*g(u^*A)).$$

Calculations show that (φ, γ) is a fibered morphism, and that we obtain bijections

$$\mathsf{Psfun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathsf{Fibgr}(\mathfrak{a}, \mathfrak{b}),$$

defining a functor $\mathsf{Psfun} \longrightarrow \mathsf{Fibgr}$. Let us now start with $(\mathcal{A}, l, 1, z)$, the associated $(\mathfrak{a}, \mu, id, \delta)$, and its associated $(\mathcal{A}', l', 1', z')$. It then suffices to construct an isomorphism $\mathcal{A} \longrightarrow \mathcal{A}'$ in Psfun. We first define a map $g : \mathcal{A} \longrightarrow \mathcal{A}'$ on the module level by

$$g^{A,B} = m(z^B, -) : \mathcal{A}(U)(A, B) \longrightarrow \mathcal{A}(U)(A, 1^*(B) = \mathcal{A}'(U)(A, B).$$

Since z^B is an isomorphism in $\mathcal{A}(U)$, this is certainly an isomorphism. Next we define

$$\alpha^A = 1'^{u^*A} = z^{u^*A} \in \mathcal{A}(V)(u^*A, 1^*u^*A) = \mathcal{A}'(U)(u^*A, u^*A).$$

It remains to check that (g, α) defines a morphism of pseudofunctors. We note that $g(1^A) = z^A = id^A = 1'^A$. We compute $m'(g(a), g(b)) = m(c, 1^*(m(z, a)), m(z, b)) = m(c, 1^*(z), 1^*(a), z, b) = m(1^*(a), z, b) = m(z, a, b) = g(m(a, b))$. It follows from equation (9) and another computation that $f' = m(z, c^{u,1}, f)$, and consequently $m'(\alpha, f'g) = f'g = m(z, c^{u,1}, f(z), f) = m(z, f) = gf = m'(gf, \alpha)$. Next, $m'(g(c), \alpha, \alpha) = g(c) = m(z, c)$. To prove that this expression equals c', we verify equation (10). This amounts to the fact that $m(c, m(z, c)) = m(m(c, z), c) = c$. Finally, we have to check that $g(z) = m(z, z)$ satisfies the relation for z' in equation (8). Indeed, we have $\mu(\delta, m(z, z)) = m(c, m(z, z)) = m(c, z, z) = z = id$.

It is not hard to see that the functors we defined between Psfun and Fibgr restrict to functors between Psfun_0 and Fibgr_0, and that the restrictions become inverse isomorphisms of categories. ∎

Note that the composition $\mathsf{Psfun} \longrightarrow \mathsf{Fibgr} \longrightarrow \mathsf{Psfun}$ mingles up the roles of local and global identities. By restricting to pseudofunctors in Psfun_0, this phenomenon becomes invisible. For our applications to deformation theory in the next section, we note the following precise statement.

Proposition 2.23. Let $(\mathcal{A}, 1, 1)$ be a lax quiver with local preidentity and trivial global preidentity, and let $(\mathfrak{a}, id, \delta)$ be the corresponding graded quiver with a preidentity and a prefibered structure. The isomorphism $\mathsf{Psfun}_0 \cong \mathsf{Fibgr}_0$ induces a 1-1 correspondence between

- lax structures l on $\mathcal{A}$ making $(\mathcal{A}, l, 1, 1)$ into a pseudofunctor;
- compositions μ on $\mathfrak{a}$ making $(\mathfrak{a}, \mu, id, \delta)$ into a fibered-graded category. □

Proof. This is clear. ■

3 Hochschild Cohomology and Deformations

3.1 Graded colored operads

In this section, we introduce *graded* colored operads of k-modules, and we show that they give rise to natural brace algebras and B_∞-algebras. All the notions and proofs in this section are immediate adaptations of [5], and most details are left to the reader. Let us start by recalling the definition of a colored operad.

Definition 3.1. Consider a set X of "colors" and a symmetric monoidal category $\mathcal{M}$. An (a-symmetric) X-colored $\mathcal{M}$-operad $\mathcal{O}$ consists of the datum, for every tuple $\underline{c} = (c_n, \dots, c_0; c)$ of colors $c_i, c \in X$, of an $\mathcal{M}$-object $\mathcal{O}(\underline{c})$ of "operations," together with "compositions"

$$\mathcal{O}(\underline{c}) \otimes \otimes_{i=0}^{n} \mathcal{O}\big(c_i^{k_i}, \dots c_i^0; c_i\big) \longrightarrow \mathcal{O}\big(c_n^{k_n}, \dots, c_0^0; c\big)$$

and "identities" $1_c \in \mathcal{O}(c; c)$, satisfying the operadic associativity and identity relations. Morphisms of colored operads are defined in the obvious way. □

Example 3.2. Let $\mathcal{V}$ be a Set-quiver with *associative* multiplications

$$m^{U,V,W} : \mathcal{V}(V, W) \times \mathcal{V}(U, V) \longrightarrow \mathcal{V}(U, W).$$

We define a colored Set-operad $\mathcal{O}_{\mathcal{V}}$ with a color set $X = \mathrm{Mor}(\mathcal{V})$ and

$$\mathcal{O}_{\mathcal{V}}(v_n, \dots, v_0; v) = \begin{cases} \{*\} & \text{if } \; v_n v_{n-1} \dots v_0 = v \\ \varnothing & \text{else} \end{cases}$$

and no 0-ary operations. If, moreover, $\mathcal{V}$ has *identities* $1^U \in \mathcal{V}(U, U)$ for m (i.e. if $\mathcal{V}$ is a category), then we define $\mathcal{O}_{\mathcal{V},1}$ where the only difference with $\mathcal{O}_{\mathcal{V}}$ is the existence of 0-ary operations

$$\mathcal{O}_{\mathcal{V},1}(1^U) = \{*\}.$$

There is an obvious morphism of operads $\mathcal{O}_{\mathcal{V}} \longrightarrow \mathcal{O}_{\mathcal{V},1}$. □

Example 3.3. Suppose $\mathcal{O}$ is an X-$\mathcal{M}$-operad and $F : \mathcal{M} \longrightarrow \mathcal{M}'$ is a monoidal functor. We define an X-$\mathcal{M}'$-operad $F\mathcal{O}$ with

$$(F\mathcal{O})(c_n, \dots c_0; c) = F(\mathcal{O}(c_n, \dots c_0; c))$$

and all operations induced by those of $\mathcal{O}$. In particular, for the free k-module functor $F : \mathsf{Set} \longrightarrow \mathsf{Mod}(k)$, we denote $k\mathcal{O} = F\mathcal{O}$. □

For our purpose, we need the notion of a *graded* colored operad.

Definition 3.4. Let $\mathcal{X}$ be an X-Set-operad. An $\mathcal{X}$-graded $\mathcal{M}$-operad $\mathcal{O}$ consists of, for every tuple $(c_n, \dots, c_0, c)$ of colors in X and for every $\xi \in \mathcal{X}(c_n, \dots c_0, c)$, an $\mathcal{M}$-object

$$\mathcal{O}_{\xi}(c_n, \dots c_0; c),$$

together with morphisms

$$\mathcal{O}_{\xi}(\underline{c}) \otimes \otimes_{i=0}^{n} \mathcal{O}_{\xi_i}\left(c_i^{k_i}, \dots c_i^0; c_i\right) \longrightarrow \mathcal{O}_{\xi(\xi_n, \dots, \xi_0)}\left(c_n^{k_n}, \dots, c_0^0; c\right)$$

and "identities" $1_c \in \mathcal{O}_1(c; c)$, satisfying "operadic" associativity and identity relations. Morphisms of graded colored operads are defined in the obvious way. □

Remark 3.5. If $\mathcal{M} = \mathsf{Set}$, then an $\mathcal{X}$-Set-operad corresponds precisely to an X-Set-operad $\mathcal{O}$, together with a morphism of X-Set-operads $\mathcal{O} \longrightarrow \mathcal{X}$. □

Example 3.6. Let $\mathcal{X}$ be an X-Set-operad. We define an $\mathcal{X}$-Set-operad $\tilde{\mathcal{X}}$ with

$$\tilde{\mathcal{X}}_{\xi}(c_n, \dots, c_0; c) = \{*\}$$

for every $\xi \in \mathcal{X}(c_n, \dots, c_0; c)$. □

Let $\mathcal{X}$ be an X-Set-operad. An *$\mathcal{X}$-graded $\mathcal{M}$-object* is an X-indexed collection $A = (A(c))_{c \in X}$ of $\mathcal{M}$-objects $A(c)$. To an $\mathcal{X}$-$\mathcal{C}$-object A, we associate the *$\mathcal{X}$-$\mathcal{M}$ endomorphism operad $\mathcal{E}nd(A)$* with

$$\mathcal{E}nd(A)_{\xi}(c_n, \dots c_0; c) = \mathcal{M}(A(c_n) \otimes \cdots \otimes A(c_0), A(c))$$

and where all the operations are given by substitution.

Definition 3.7. Let $\mathcal{O}$ be an $\mathcal{X}$-$\mathcal{M}$-operad. An $\mathcal{O}$-algebra is an $\mathcal{X}$-$\mathcal{M}$-object A, together with a morphism of $\mathcal{X}$-colored operads

$$\mathcal{O} \longrightarrow \mathcal{E}nd(A).$$

□

From now on, we take $\mathcal{M} = \mathsf{Mod}(k)$.
For an $\mathcal{X}$-graded operad $\mathcal{O}$, we consider the $\mathbb{Z}$-graded module $\mathbf{C}(\mathcal{O})$ with

$$\mathbf{C}(\mathcal{O})^n = \prod_{\substack{c_{n-1}, \dots c_0; c \in X \\ \xi \in \mathcal{X}(c_{n-1}, \dots c_0; c)}} \mathcal{O}_{\xi}(c_{n-1}, \dots c_0; c).$$

We also consider the shifted object $\mathbf{C}(\mathcal{O})[1]$ with $\mathbf{C}(\mathcal{O})[1]^n = \mathbf{C}(\mathcal{O})^{n+1}$, and for $\phi \in \mathbf{C}(\mathcal{O})^n$ we put $|\phi| = n - 1$, the degree of ϕ in $\mathbf{C}(\mathcal{O})[1]$.

We obtain natural morphisms

$$\mathbf{C}(\mathcal{O})^n \otimes \mathbf{C}(\mathcal{O})^{n_1} \otimes \cdots \otimes \mathbf{C}(\mathcal{O})^{n_k} \longrightarrow \mathbf{C}(\mathcal{O})^{n+n_1+\cdots+n_k-k},$$

defined by the components, for $\xi \in \mathcal{X}(c_{n-1}, \dots, c_0; c)$,

$$\phi\{\phi_1, \dots, \phi_k\}^{\xi} = \sum_{\xi = \xi'(1, \dots, \xi_1, \dots, 1, \xi_n)} (-1)^{\epsilon} \phi^{\xi'}\left(1, \dots, \phi_1^{\xi_1}, \dots, 1, \phi_k^{\xi_k}\right) \tag{11}$$

with $\epsilon = \sum_{p=1}^{k} |\phi_p| i_p$, where i_p is the number of inputs in front of ϕ_p.

We recall the definition of a brace algebra.

Definition 3.8. For a $\mathbb{Z}$-graded k-module V, the structure of brace algebra on V consists in the datum of (degree zero) operations

$$V^{\otimes n+1} \longrightarrow V : (x, x_1, \dots x_n) \longmapsto x\{x_1, \dots x_n\},$$

satisfying the relation

$$x\{x_1, \dots x_m\}\{y_1, \dots y_n\} = \sum (-1)^{\alpha} x\{y_1, \dots, x_1\{y_{i_1}, \dots\}, y_{j_1} \dots, x_m\{y_{i_m}, \dots\}, y_{j_m}, \dots y_n\},$$

where $\alpha = \sum_{k=1}^{m} |x_k| \sum_{l=1}^{i_k-1} |y_l|$. The dot product of the brace algebra is by definition $x \bullet y = x\{y\}$. The associated Lie bracket of the brace algebra is

$$[x, y] = x \bullet y - (-1)^{|x||y|} y \bullet x.$$

An element $1^V \in V^0$ is an identity for V if for any $\phi \in V^n$, we have $1^V \bullet \phi = \phi$ and $\phi\{(1^V)^{\otimes n+1}\} = \phi$. □

Proposition 3.9. Let $\mathcal{O}$ be an $\mathcal{X}$-graded operad. Then $\mathbf{C}(\mathcal{O})[1]$ is a brace algebra for the brace operations (11). The identity elements $1_c \in \mathcal{O}_1(c; c)$ define an element in $\mathcal{C}(\mathcal{O})^1$ which is an identity for the brace algebra. □

Next we take $\mathcal{X} = \mathcal{O}_{\mathcal{V}}$ for a small category $\mathcal{V}$. It is clear that for an operation $\mu \in \mathbf{C}(\mathcal{O})^2$, the equation $\mu \bullet \mu = 0$ expresses precisely that μ is associative.

Proposition 3.10. Let $\mathcal{O}$ be an $\mathcal{O}_{\mathcal{V}}$-graded operad. There are bijections between

(1) maps of $\mathcal{O}_{\mathcal{V}}$-graded operads $k\tilde{\mathcal{O}}_{\mathcal{V}} \longrightarrow \mathcal{O}$;
(2) elements $\mu \in \mathbf{C}(\mathcal{O})^2$ with $\mu \bullet \mu = 0$;
(3) structures of homotopy G-algebra [5, Definition 2] on the brace algebra $\mathbf{C}(\mathcal{O})[1]$. □

As explained in detail in [12], a homotopy G-algebra is a special case of a B_∞-algebra [6]. In particular, in the case of Proposition 3.10, $\mathbf{C}(\mathcal{O})[1]$ becomes a dg Lie algebra when equipped with the differential $d = [\mu, -]$, and is endowed with a cup product $\phi \cup \psi = \mu\{\phi, \psi\}$ (see also [7]).

3.2 Hochschild cohomology of map-graded categories

Next we will apply the results of the previous section to map-graded categories. Consider a Set-quiver $\mathcal{U}$ with associative multiplications, and for every $U \in \mathcal{U}$ a set $\mathfrak{a}_U$. From this we build the Set-quiver $\mathcal{V}$ with associative multiplications with $\mathrm{Ob}(\mathcal{V}) = \coprod_{U \in \mathcal{U}} \mathfrak{a}_U$ and for $A \in \mathfrak{a}_V$, $B \in \mathfrak{a}_U$, $\mathcal{V}(A, B) = \mathcal{U}(V, U)$. Then $\mathcal{O}_{\mathcal{V}}$ and $\mathcal{O}_{\mathcal{V},1}$ are X-Set-operads with colors in X given by

$$\begin{tikzcd} A & B \\ V \arrow[r, "u"'] & U \end{tikzcd}$$

with $A \in \mathfrak{a}_V$, $B \in \mathfrak{b}_U$, and $u : V \longrightarrow U$ in $\mathcal{U}$. An $\mathcal{O}_{\mathcal{V}}$-$\mathsf{Mod}(k)$-object $\mathfrak{a}$ consists in the datum, for every color (A, u, B) as above, of a k-module

$$\mathfrak{a}_u(A, B),$$

in other words, this is precisely a $\mathcal{U}$-graded k-quiver. The structure of $k\tilde{\mathcal{O}}_{\mathcal{V}}$-algebra on $\mathfrak{a}$ corresponds to a composition structure μ, making $(\mathfrak{a}, \mu)$ into an associative $\mathcal{U}$-graded quiver. If $\mathcal{U}$ is a category, then the structure of $k\tilde{\mathcal{O}}_{\mathcal{V},1}$-algebra on $\mathfrak{a}$ corresponds to the additional datum of an identity structure id, making $(\mathfrak{a}, \mu, id)$ into a $\mathcal{U}$-graded category.

Definition 3.11. For a $\mathcal{U}$-graded quiver $\mathfrak{a}$, the *Hochschild brace algebra* $\mathbf{C}(\mathfrak{a})$ of $\mathfrak{a}$ is the brace algebra $\mathbf{C}(\mathcal{E}nd(\mathfrak{a}))$ of Proposition 3.9. If $(\mathfrak{a}, \mu)$ is an associative $\mathcal{U}$-graded quiver, the *Hochschild complex* of $\mathfrak{a}$ is $\mathbf{C}(\mathfrak{a})$ with the B_∞-structure of Proposition 3.10. □

Concretely,

$$\mathbf{C}^n(\mathfrak{a}) = \prod_{\substack{u_{n-1},\dots,u_0 \\ A_0,\dots A_n}} \mathrm{Hom}_k\big(\otimes_{i=0}^{n-1} \mathfrak{a}_{u_i}(A_i, A_{i+1}), \mathfrak{a}_{u_{n-1}\dots u_0}(A_0, A_n)\big)$$

and the B_∞-structure on $\mathbf{C}(\mathfrak{a})$ is analogous to the structure on the Hochschild complex of an associative algebra. The *Hochschild cohomology* of $\mathfrak{a}$ is by definition the cohomology of the complex $\mathbf{C}(\mathfrak{a})$.

From now on, $\mathfrak{a}$ is a *$\mathcal{U}$-graded category*. As in the case of an algebra, Hochschild cohomology can be expressed as an Ext in the category of bimodules. For this, we consider the $\mathfrak{a}$-bimodule $\mathfrak{a}$ with

$$\mathfrak{a}(A, u, B) = \mathfrak{a}_u(A, B),$$

as well as its submodule $\underline{k}$ with $\underline{k}_1(A, A) = k1_A$ and $\underline{k}_u(A, B) = 0$ if $(A, u, B) \neq (A, 1, A)$ for some A, and the quotient module $\bar{\mathfrak{a}} = \mathfrak{a}/\underline{k}$.

With the isomorphism of Proposition 2.11 in mind, consider for $u : V \longrightarrow U$, $A \in \mathfrak{a}_V$, and $B \in \mathfrak{b}_U$ the image $P_{A,u,B}$ of (A, u, B) under

$$\mathfrak{a}^{\mathrm{op}} \otimes_{\mathcal{U}} \mathfrak{a} \longrightarrow \mathsf{Mod}(\mathfrak{a}^{\mathrm{op}} \otimes_{\mathcal{U}} \mathfrak{a}) \cong \mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a}, \mathfrak{a}).$$

Next, consider the bimodules

$$B^n_{A,u,B} = P_{A,u,B} \otimes_k \bigoplus_{\substack{u=u_{n-1}\dots u_0 \\ A=A_0,\dots,A_n=B}} \otimes_{i=0}^{n-1} \mathfrak{a}_{u_i}(A_i, A_{i+1})$$

and

$$\bar{B}^n_{A,u,B} = P_{A,u,B} \otimes_k \bigoplus_{\substack{u=u_{n-1}\dots u_0 \\ A=A_0,\dots,A_n=B}} \otimes_{i=0}^{n-1} \bar{\mathfrak{a}}_{u_i}(A_i, A_{i+1}).$$

Lemma 3.12. We obtain projective resolutions

$$\dots \longrightarrow \bigoplus_{A,u,B} B^n_{A,u,B} \longrightarrow \dots \longrightarrow \bigoplus_{A,u,B} B^1_{A,u,B} \longrightarrow \bigoplus_{A} P_{A,1,A} \longrightarrow \mathfrak{a} \longrightarrow 0,$$

and likewise with $B^n_{A,u,B}$ replaced by $\bar{B}^n_{A,u,B}$, where the value

$$\begin{array}{c} \bigoplus_{\substack{v=zu_{n-1}\dots u_0 w \\ A_0,\dots,A_n}} \mathfrak{a}_z(A_n, D) \otimes \bigotimes_{i=0}^{n-1} \mathfrak{a}_{u_i}(A_i, A_{i+1}) \otimes \mathfrak{a}_w(C, A_0) \\ \Big\downarrow{\scriptstyle d_n} \\ \bigoplus_{\substack{v=z'v_{n-2}\dots v_0 w' \\ B_0,\dots,B_n}} \mathfrak{a}_{z'}(B_{n-1}, D) \otimes \bigotimes_{i=0}^{n-2} \mathfrak{a}_{v_i}(B_i, B_{i+1}) \otimes \mathfrak{a}_{w'}(C, B_0) \end{array}$$

of the differential on a fixed (C, v, D) is given by

$$d_n(a_n, \dots, a_0, a_{-1}) = \sum_{i=0}^{n} (-1)^i (a_n, \dots, \mu(a_i, a_{i-1}), \dots, a_{-1}).$$

□

Proof. Like in the classical proof, one can easily write down a k-linear 0-homotopy. Note that since te differential respects $\bar{B}^n$, we also obtain a projective resolution with B^n replaced by $\bar{B}^n$. ■

The projective resolutions of Lemma 3.12 are the *bar resolution* $B(\mathfrak{a})$ and the *normalized bar resolution* $\bar{B}(\mathfrak{a})$ of $\mathfrak{a}$.

Proposition 3.13. Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category. There is an isomorphism of complexes

$$\mathbf{C}(\mathfrak{a}) \cong \mathrm{Hom}_{\mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a})}(B(\mathfrak{a}), \mathfrak{a}).$$

The canonical epimorphism $B(\mathfrak{a}) \longrightarrow \bar{B}(\mathfrak{a})$ induces a subcomplex

$$\bar{\mathbf{C}}(\mathfrak{a}) \longrightarrow \mathbf{C}(\mathfrak{a})$$

which is a subhomotopy $G-(hencesubB_{\infty}-)$algebra. Moreover, the inclusion is a quasi-isomorphism and the cohomology of both sides is given by

$$HH^n(\mathfrak{a}) = \mathrm{Ext}^n_{\mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a})}(\mathfrak{a}, \mathfrak{a}).$$

□

Proof. Like in the classical situation, this is easy. ■

Note that the identity element $(1_{\mathfrak{a}_u(A,B)} \in \mathrm{Hom}_k(\mathfrak{a}_u(A, B), \mathfrak{a}_u(A, B))) \in \mathbf{C}(\mathfrak{a})^1$ of the brace algebra $\mathbf{C}(\mathfrak{a})[1]$ does not belong to $\bar{\mathbf{C}}(\mathfrak{a})$, and neither does the element $\mu \in \mathbf{C}(\mathfrak{a})^2$.

3.3 Deformations of map-graded categories

As in the case of associative algebras, the Hochschild complex $\mathbf{C}(\mathfrak{a})$ of an associative $\mathcal{U}$-graded quiver $\mathfrak{a}$ controls the algebraic deformation theory of its multiplication $\mu \in \mathbf{C}^2(\mathfrak{a})$.

From now on, let k be a field and $\mathsf{Art}(k)$ the category of artin local k-algebras $(R, \mathfrak{m})$ with residue field k. We will consider and compare a number of deformation pseudofunctors

$$\mathsf{Art}(k) \longrightarrow \mathsf{Gd}$$

with values in groupoids. Deformation pseudofunctors arise in the following way.

Definition 3.14. Let $F : \mathsf{Art}(k) \longrightarrow \mathsf{Cat}$ be a pseudofunctor and consider an object $X \in F(k)$. An *R-deformation* of X is an object $Y \in F(R)$, together with an $F(k)$-isomorphism $f : F(R \longrightarrow k)(Y) \longrightarrow X$. An *equivalence* between deformations (Y, f) and (Y', f') is an $F(R)$-isomorphism $g : Y \longrightarrow Y'$ with $f' F(R \longrightarrow k)(g) = f$. The R-deformations of X and their equivalences form a *deformation groupoid* $\mathrm{Def}_{F,X}(R)$, giving rise to a *deformation pseudofunctor* $\mathrm{Def}_{F,X} : \mathsf{Art}(k) \longrightarrow \mathsf{Gd}$. $\square$

A $\mathcal{U}$-graded R-category $\mathfrak{a}$ is called flat if all the $\mathfrak{a}_u(A, B)$ are flat R-modules (or, equivalently, since R is in $\mathsf{Art}(k)$, free R-modules). We denote the full subcategory of R-flat $\mathcal{U}$-graded categories by $\mathsf{Mapgr}^{\mathrm{fl}}(R)$. We will look at the pseudofunctors

$$\mathsf{Mapgr}^{\mathrm{fl}} : \mathsf{Art}(k)^{\mathrm{op}} \longrightarrow \mathsf{Cat} : R \longmapsto \mathsf{Mapgr}^{\mathrm{fl}}(R)$$

and

$$\mathsf{Fibgr}^{\mathrm{fl}} : \mathsf{Art}(k)^{\mathrm{op}} \longrightarrow \mathsf{Cat} : R \longmapsto \mathsf{Fibgr}^{\mathrm{fl}}(R),$$

where a morphism $R' \longrightarrow R$ in $\mathsf{Art}(k)$ corresponds to $R \otimes_{R'} - : \mathsf{Mapgr}^{\mathrm{fl}}(R') \longrightarrow \mathsf{Mapgr}^{\mathrm{fl}}(R)$ and similarly for $\mathsf{Fibgr}^{\mathrm{fl}}$. For a $\mathcal{U}$-graded k-category $\mathfrak{a}$, we consider the deformation pseudofunctors $\mathrm{Def}^{\mathfrak{a}} = \mathrm{Def}_{\mathsf{Mapgr}^{\mathrm{fl}},\mathfrak{a}}$ and $\mathrm{Def}^{\mathfrak{a}}_{\mathrm{fib}} = \mathrm{Def}_{\mathsf{Fibgr}^{\mathrm{fl}},\mathfrak{a}}$.

In order to relate deformations to the Hochschild complex, we introduce three more deformation pseudofunctors.

Definition 3.15. Let $(\mathfrak{a}, \mu)$ be an associative $\mathcal{U}$-graded k-quiver.

(1) An *R-deformation of μ* is a structure $\bar{\mu}$ of associative $\mathcal{U}$-graded R-quiver on $R \otimes_k \mathfrak{a} = \mathfrak{a} \oplus (\mathfrak{m} \otimes_k \mathfrak{a})$ that reduces to μ on $\mathfrak{a}$.

(2) An *equivalence* between deformations $\bar{\mu}$ and $\bar{\mu}'$ is an isomorphism $\varphi : (R \otimes_k \mathfrak{a}, \bar{\mu}) \longrightarrow (R \otimes_k \mathfrak{a}, \bar{\mu}')$ that reduces to $1 : (\mathfrak{a}, \mu) \longrightarrow (\mathfrak{a}, \mu)$.

(3) The associated deformation pseudofunctor is denoted by $\mathrm{Def}^{\mu}_{\mathfrak{a}}$.

Let $(\mathfrak{a}, \mu, id)$ be a $\mathcal{U}$-graded k-category.

(1) An *R-deformation of μ respecting id* is a composition structure $\bar{\mu}$ on $R \otimes_k \mathfrak{a}$ that reduces to μ on $\mathfrak{a}$, such that $(R \otimes_k \mathfrak{a}, \bar{\mu}, id)$ becomes a $\mathcal{U}$-graded R-category.

(2) An *equivalence* between deformations $\bar{\mu}$ and $\bar{\mu}'$ is an isomorphism $\varphi : (R \otimes_k \mathfrak{a}, \bar{\mu}, id) \longrightarrow (R \otimes_k \mathfrak{a}, \bar{\mu}', id)$ that reduces to $1 : (\mathfrak{a}, \mu, id) \longrightarrow (\mathfrak{a}, \mu, id)$.

(3) The associated deformation pseudofunctor is denoted by $\mathrm{Def}^{\mu}_{\mathfrak{a},id}$.

Let $(\mathfrak{a}, \mu, id, \delta)$ be a fibered $\mathcal{U}$-graded k-category.

(1) A *fibered R-deformation of* μ is a composition structure $\bar{\mu}$ on $R \otimes_k \mathfrak{a}$ that reduces to μ on $\mathfrak{a}$, such that $(R \otimes_k \mathfrak{a}, \bar{\mu}, id, \delta)$ becomes a fibered $\mathcal{U}$-graded R-category.

(2) An *equivalence* of fibered deformations $\bar{\mu}$ and $\bar{\mu}'$ is an isomorphism (φ, γ) : $(R \otimes_k \mathfrak{a}, \bar{\mu}, id, \delta) \longrightarrow (R \otimes_k \mathfrak{a}, \bar{\mu}', id, \delta)$ that reduces to $(1, 1) : (\mathfrak{a}, \mu, id, \delta) \longrightarrow (\mathfrak{a}, \mu, id, \delta)$.

(3) The associated deformation pseudofunctor is denoted by $\mathrm{Def}^{\mu}_{\mathfrak{a},id,\delta}$. □

Of course, dg Lie algebras are another source of deformation pseudofunctors. We briefly recall the idea. Let L be a nilpotent dg Lie algebra. We denote $\overline{MC}(L) = \{x \in L^1 \mid d(x) + 1/2[x, x] = 0\}$, the set of solutions of the Maurer Cartan equation. The Lie algebra L^0 has an infinitesimal action ρ on $\overline{MC}(L)$ given by $\rho(y)(x) = d(y) + [x, y]$ for $y \in L^0$, $x \in L^1$, that integrates to an action of $G = \exp(L^0)$ on $\overline{MC}(L)$. The Maurer Cartan groupoid (also called the Deligne groupoid) $MC(L)$ of L is the natural groupoid associated to this action.

Now let L be an arbitrary dg Lie algebra. For every $(R, \mathfrak{m}) \in \mathsf{Art}(k)$, we obtain a nilpotent dg Lie algebra $\mathfrak{m} \otimes_k L$, and we obtain a deformation pseudofunctor Def_L with

$$\mathrm{Def}_L(R) = MC(\mathfrak{m} \otimes_k L).$$

Proposition 3.16. Let $(\mathfrak{a}, \mu)$ be an associative $\mathcal{U}$-graded quiver. There is an isomorphism of deformation pseudofunctors

$$\mathrm{Def}_{\mathbf{C}(\mathfrak{a})} \longrightarrow \mathrm{Def}^{\mu}_{\mathfrak{a}}.$$ □

Proof. By definition, an R-deformation of μ is determined by an element $\mu' \in \mathfrak{m} \otimes_k \mathbf{C}(\mathfrak{a})$ via the injection $\mathfrak{m} \otimes_k \mathbf{C}(\mathfrak{a}) \longrightarrow \mathbf{C}(R \otimes_k \mathfrak{a}) : \mu' \longmapsto \mu + \mu'$. The fact that μ' defines a deformation is equivalent to $(\mu + \mu') \bullet (\mu + \mu') = 0$, or in terms of $\mathfrak{m} \otimes_k \mathbf{C}(\mathfrak{a})$, to $\mu \bullet \mu' + \mu' \bullet \mu + \mu' \bullet \mu' = 0$, which translates precisely into the Maurer Cartan equation. For an arbitrary $\varphi \in \mathfrak{m} \otimes_k \mathbf{C}^1(\mathfrak{a})$, we obtain an isomorphism of quivers $\exp(\varphi) : R \otimes_k \mathfrak{a} \longrightarrow R \otimes_k \mathfrak{a}$, yielding actions on $\mathbf{C}(R \otimes_k \mathfrak{a})$ and $\mathfrak{m} \otimes_k \mathbf{C}(\mathfrak{a})$ by "transport of structure." As in the case of associative algebras, the differential of this action is seen to coincide with the one defining the Maurer Cartan groupoid. ■

For a $\mathcal{U}$-graded category $\mathfrak{a}$, we are interested in deformations preserving the identity id of $\mathfrak{a}$.

Proposition 3.17. Let $(\mathfrak{a}, \mu, id)$ be a $\mathcal{U}$-graded category. There is an isomorphism of deformation pseudofunctors

$$\mathrm{Def}_{\bar{\mathbf{C}}(\mathfrak{a})} \longrightarrow \mathrm{Def}^{\mu}_{\mathfrak{a},id}. \qquad \square$$

Proof. Starting from the isomorphism of Proposition 3.16, it is immediate that $\bar{\mu} = \mu + \mu'$ respects id if and only if $(\mu + \mu')(id, a) = a = (\mu + \mu')(a, id) = 0$, if and only if $\mu'(id, a) = 0 = \mu'(a, id)$, if and only if μ' belongs to $\mathfrak{m} \otimes_k \bar{\mathbf{C}}(\mathfrak{a})$. ■

Proposition 3.18. Let $(\mathfrak{a}, \mu, id)$ be a $\mathcal{U}$-graded category. The natural map

$$\mathrm{Def}^{\mu}_{\mathfrak{a},id} \longrightarrow \mathrm{Def}^{\mu}_{\mathfrak{a}}$$

is an equivalence. □

Proof. According to the quasi-isomorphism theorem, the quasi-isomorphism $\bar{\mathbf{C}}(\mathfrak{a}) \longrightarrow \mathbf{C}(\mathfrak{a})$ of dg Lie algebras induces an equivalence of deformation functors $\mathrm{Def}_{\bar{\mathbf{C}}(\mathfrak{a})} \longrightarrow \mathrm{Def}_{\mathbf{C}(\mathfrak{a})}$. Via the isomorphisms of Propositions 3.16 and 3.17, this equivalence corresponds precisely to the natural $\mathrm{Def}^{\mu}_{\mathfrak{a},id} \longrightarrow \mathrm{Def}^{\mu}_{\mathfrak{a}}$. ■

Proposition 3.19. Let $(\mathfrak{a}, \mu, id)$ be a $\mathcal{U}$-graded category. The natural map

$$\mathrm{Def}^{\mu}_{\mathfrak{a},id} \longrightarrow \mathrm{Def}^{\mathfrak{a}}$$

is an equivalence. □

Proof. The map is fully faithful by definition. Suppose $\bar{\mathfrak{b}}$, together with the isomorphism $\varphi : \mathfrak{b} = k \otimes_R \bar{\mathfrak{b}} \longrightarrow \mathfrak{a}$ is an R-deformation of $\mathfrak{a}$. By flatness we may suppose, by changing $\bar{\mathfrak{b}}$ up to equivalence of deformations, that $\bar{\mathfrak{b}}_u(A, B) = R \otimes_k \mathfrak{b}_u(A, B)$. We obtain an isomorphism $R \otimes_k \varphi : \bar{\mathfrak{b}} \longrightarrow R \otimes_k \mathfrak{a}$ of $\mathcal{U}$-R-quivers, making it possible to transport the composition of $\bar{\mathfrak{b}}$ to a composition $\bar{\mu}$ deforming μ. By Proposition 3.18, $\bar{\mu}$ is equivalent to a deformation $\bar{\mu}'$ of μ respecting id, and this equivalence of deformations necessarily preserves the identities. ■

Proposition 3.20. Let $(\mathfrak{a}, \mu, id, \delta)$ be a fibered $\mathcal{U}$-graded category. The natural map

$$\mathrm{Def}^{\mu}_{\mathfrak{a},id,\delta} \longrightarrow \mathrm{Def}^{\mu}_{\mathfrak{a},id}$$

is an isomorphism. □

Proof. Let $\bar{\mu}$ be an R-deformation of μ respecting id. When we consider the elements $\delta^{u,A} \in \mathfrak{a}_u(A,B) \subseteq R \otimes_k \mathfrak{a}_u(A,B)$, the relevant morphisms $\bar{\mu}(\delta,-)$ lift the morphisms $\mu(\delta,-)$, whence they remain isomorphisms and the $\delta^{u,A}$ remain cartesian in $(R \otimes_k \mathfrak{a}, \bar{\mu})$. Consequently, $\bar{\mu}$ is a fibered R-deformation of μ. Now suppose $\varphi : (R \otimes_k \mathfrak{a}, \bar{\mu}, id) \longrightarrow (R \otimes_k \mathfrak{a}, \bar{\mu}', id)$ is an equivalence of deformations of μ respecting id. According to Proposition 2.7, there is a unique isomorphism of fibered categories $(\varphi, \gamma) : (R \otimes_k \mathfrak{a}, \bar{\mu}, id, \delta) \longrightarrow (R \otimes_k \mathfrak{a}, \bar{\mu}', id, \delta)$, which is readily seen to deform the identity morphism. ■

Proposition 3.21. Let $(\mathfrak{a}, \mu, id, \delta)$ be a fibered $\mathcal{U}$-graded category. The natural map

$$\mathrm{Def}^{\mu}_{\mathfrak{a},id,\delta} \longrightarrow \mathrm{Def}^{\mathfrak{a}}_{\mathrm{fib}}$$

is an equivalence. □

Proof. The map is fully faithful by definition. By Proposition 3.19, a fibered deformation $\bar{\mathfrak{b}}$ of $\mathfrak{a}$ is isomorphic as a graded deformation to a deformation $(R \otimes_k \mathfrak{a}, \bar{\mu}, id)$. Hence, by Propositions 3.20 and 2.7, it is isomorphic as a fibered deformation to $(R \otimes_k \mathfrak{a}, \bar{\mu}, id, \delta)$. ■

3.4 Deformations of Pseudofunctors

Next we apply the results of the previous section to deformations of pseudofunctors (with a special interest in presheaves).

Proposition 3.22. Consider pseudofunctors $F, G : \mathsf{Art}(k) \longrightarrow \mathsf{Cat}$ and a morphism of pseudofunctors $\Psi : F \longrightarrow G$. Consider an object $X \in F(k)$ and its image $Y = \Psi(k)(X) \in G(k)$. If Ψ is an equivalence (respectively, an isomorphism), then so is the induced $\mathrm{Def}_{F,X} \longrightarrow \mathrm{Def}_{G,Y}$. □

Proof. This is easily seen. ■

A $\mathcal{U}$-R-pseudofunctor $\mathcal{A}$ is called *flat* if all the modules $\mathcal{A}(U)(A, B)$ are flat over R.

Proposition 3.23. There is a square of pseudofunctors $\mathsf{Art}(k) \longrightarrow \mathsf{Cat}$,

$$\begin{tikzcd} F = \mathsf{Psfun}^{\mathrm{fl}} \arrow[r] & \mathsf{Fibgr}^{\mathrm{fl}} = G \\ F_0 = \mathsf{Psfun}_0^{\mathrm{fl}} \arrow[u] \arrow[r] & \mathsf{Fibgr}_0^{\mathrm{fl}} = G_0 \arrow[u] \end{tikzcd}$$

in which all the arrows are equivalences, and the lower arrow is an isomorphism. It induces the following equivalences of deformation pseudofunctors:

(1) $\mathrm{Def}_0^{\mathcal{A}} = \mathrm{Def}_{F_0,\mathcal{A}} \longrightarrow \mathrm{Def}_{F,\mathcal{A}} = \mathrm{Def}^{\mathcal{A}}$ for $\mathcal{A} \in \mathsf{Psfun}_0(k)$;
(2) $\mathrm{Def}_{\mathrm{fib},0}^{\mathfrak{a}} = \mathrm{Def}_{G_0,\mathfrak{a}} \longrightarrow \mathrm{Def}_{G,\mathfrak{a}} = \mathrm{Def}_{\mathrm{fib}}^{\mathfrak{a}}$ for $\mathfrak{a} \in \mathsf{Fibgr}_0(k)$;
(3) $\mathrm{Def}^{\mathcal{A}} = \mathrm{Def}_{F,\mathcal{A}} \longrightarrow \mathrm{Def}_{G,\mathfrak{a}} = \mathrm{Def}_{\mathrm{fib}}^{\mathfrak{a}}$ for $\mathcal{A} \in \mathsf{Psfun}(k)$;
(4) $\mathrm{Def}_0^{\mathcal{A}} = \mathrm{Def}_{F_0,\mathfrak{a}} \longrightarrow \mathrm{Def}_{G_0,\mathcal{A}} = \mathrm{Def}_{\mathrm{fib},0}^{\mathfrak{a}}$ for $\mathcal{A} \in \mathsf{Psfun}_0(k)$,

in which the last one is an isomorphism, and where $\mathfrak{a}$ is the $\mathcal{U}$-graded category associated to $\mathcal{A}$. □

Proof. This immediately follows from Theorem 2.22 and Proposition 3.22. ■

For $(\mathcal{A}, l, 1, 1) \in \mathsf{Psfun}_0(k)$, we also consider the following deformation pseudofunctor.

Definition 3.24. Consider $(\mathcal{A}, l, 1, 1) \in \mathsf{Psfun}_0(k)$.

(1) An *R-deformation* of l is a lax structure $\bar{l}$ on $\bar{\mathcal{A}} = R \otimes_k \mathcal{A} = \mathcal{A} \oplus (\mathfrak{m} \otimes_k \mathcal{A})$ that reduces to l on $\mathcal{A}$, such that $(\bar{\mathcal{A}}, \bar{l}, 1, 1)$ is a $\mathcal{U}$-R-pseudofunctor.
(2) An *equivalence* between deformations $\bar{l}$ and $\bar{l}'$ is an isomorphism (f, β) : $(\bar{\mathcal{A}}, \bar{l}, 1, 1) \longrightarrow (\bar{\mathcal{A}}, \bar{l}', 1, 1)$ that reduces to the identity $(1, 1)$ on $\mathcal{A}$.
(3) The associated deformation pseudofunctor is denoted by $\mathrm{Def}_{\mathcal{A},1,1}^{l}$. □

Proposition 3.25. Let $(\mathcal{A}, l, 1, 1)$ be a pseudofunctor in $\mathsf{Psfun}_0(k)$ and $(\mathfrak{a}, \mu, id, \delta)$ the associated fibered-graded category. There is square of deformation pseudofunctors

$$\begin{tikzcd} \mathrm{Def}_0^{\mathcal{A}} \arrow[r] & \mathrm{Def}_{\mathrm{fib},0}^{\mathfrak{a}} \\ \mathrm{Def}_{\mathcal{A},1,1}^{l} \arrow[u] \arrow[r] & \mathrm{Def}_{\mathfrak{a},id,\delta}^{\mu} \arrow[u] \end{tikzcd}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are equivalences. □

Proof. We already know from Proposition 3.23(4) that the upper arrow is an isomorphism. By Proposition 2.23, the lower arrow is an isomorphism too. It then suffices to show that the right arrow is an equivalence. For this, it suffices to note that the equivalence $\mathrm{Def}_{\mathfrak{a},id,\delta}^{\mu} \longrightarrow \mathrm{Def}_{\mathrm{fib}}^{\mathfrak{a}}$ of Proposition 3.21 factors through $\mathrm{Def}_{\mathrm{fib},0}^{\mathfrak{a}}$. ■

3.5 Hochschild cohomology of pseudofunctors

The results of the previous sections suggest the following definition.

Definition 3.26. Let $\mathcal{A}$ be a pseudofunctor with associated fibered-graded category $\mathfrak{a}$. The Hochschild complex of $\mathcal{A}$ is $\mathbf{C}(\mathcal{A}) = \mathbf{C}(\mathfrak{a})$. □

Theorem 3.27. For an arbitrary pseudofunctor $\mathcal{A}$, there is an equivalence of deformation functors

$$\mathrm{Def}^{\mathcal{A}} \cong \mathrm{Def}_{\mathbf{C}(\mathcal{A})}.$$

For a pseudofunctor $(\mathcal{A}, l, 1, 1) \in \mathsf{Psfun}_0$, there is an isomorphism of deformation functors

$$\mathrm{Def}_{\mathcal{A},1,1}^{l} \cong \mathrm{Def}_{\bar{\mathbf{C}}(\mathcal{A})}.$$ □

Proof. This follows from putting together all the results of the previous two sections. ■

Pseudofunctors over a category $\mathcal{U}$ allow a tensor product and notions of modules and bimodules, extending the same notions for presheaves. The Gerstenhaber–Schack complex $\mathbf{C}_{GS}(\mathcal{A})$ computes $H^n\mathbf{C}_{GS}(\mathcal{A}) = \mathrm{Ext}^n_{\mathsf{Bimod}_{\mathcal{U}}(\mathcal{A})}(\mathcal{A}, \mathcal{A})$, and their isomorphisms $H^n\mathbf{C}_{GS}(\mathcal{A}) \cong H^n\mathbf{C}(A)$ are a special case of their General Cohomology Comparison Theorem (GCCT) [3]. In a subsequent paper, we will prove the following "CCT for pseudofunctors."

Theorem 3.28. Let $\mathcal{A}$ be a pseudofunctor and $\mathfrak{a}$ its associated $\mathcal{U}$-graded category. There is a canonical functor

$$\mathsf{Bimod}_{\mathcal{U}}(\mathcal{A}) \longrightarrow \mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a})$$

that induces a fully faithful functor between the derived categories

$$D(\mathsf{Bimod}_{\mathcal{U}}(\mathcal{A})) \longrightarrow D(\mathsf{Bimod}_{\mathcal{U}}(\mathfrak{a})).$$

□

In particular, for a presheaf $\mathcal{A}$, our complex $\mathbf{C}(\mathcal{A}) = \mathbf{C}(\mathfrak{a})$ computes the same Hochschild cohomology as $\mathbf{C}_{GS}(\mathcal{A})$, but to understand this entire Hochschild cohomology in terms of deformations, one should view $\mathcal{A}$ as pseudofunctor (that happened to be a presheaf).

Acknowledgment

This work has been supported by the Research Foundation Flanders (FWO), where the author is a postdoctoral fellow.

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